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NAVAL RESEARCH LAB WASHINGTON DC J M WITTING ET AL.
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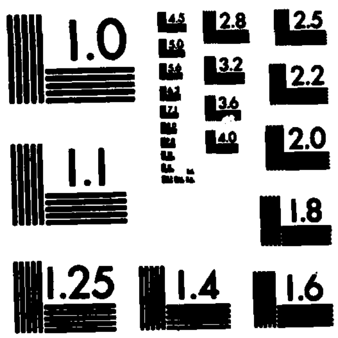
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The Euler equations for an inviscid barotropic fluid ($\rho=\rho(p)$) lead to a conservation law for the tangential component of surface velocity. The conservation law has been invoked in an approximate way in the past without having been explicitly stated. Here we derive the general form of the conservation law and, for irrotational flows, point out its relation to Bernoulli's law. We then illustrate its use in a nonlinear surface wave model and compare results with experimental profiles for solitary waves in converging/

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20. ABSTRACT (Continued)

diverging channels. The Korteweg-deVries theory originally used to interpret the experiment does not account for some features seen in the data, such as an oscillatory tail produced when a solitary wave moves through a diverging channel. By contrast, results given here, which derive from a model that explicitly conserves mass and velocity, faithfully reproduce the prominent features of the experiment.

CONTENTS

A CONSERVATION-OF-VELOCITY LAW FOR INVISCID FLUIDS

1. Introduction

Conservation laws often provide keys to the understanding of basic phenomena throughout physics. In the study of water waves a single conservation law - energy - is sufficient to derive, for example, Green's law for long linear waves (Green, 1838) and an analogous law for solitary waves in a channel of gradually varying section (see Miles, 1979). Lamb (1932) invokes two linear conservation laws (mass and velocity or momentum) in an approximate way to derive reflection coefficients for long linear waves encountering a barrier. The results agree well with the more general linear solution of Bartholomeusz (1958) when the waves are long enough. Witting (1981) invokes the two linear laws and a quadratic law to derive reflection and re-reflection coefficients for long linear waves. He shows that failure to invoke the velocity conservation law (while retaining mass conservation) removes the re-reflected wave, and alters the properties of the reflected wave.

In this brief report we show that the "conservation of velocity" law as used by Witting is a manifestation of a very general linear conservation law for inviscid water waves. For irrotational flows it is a variant of Bernoulli's law, but the general form holds for rotational flows as well. We believe that this conservation of velocity law may be novel (at least in its present form). In Section 2 we derive the law and discuss its connection to Bernoulli's law. Some applications are discussed in Section 3.

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2. Equations of Motion

A. General Derivation

We choose a coordinate system as illustrated in Fig. 1, with x horizontal, y vertical, and η (x,t) the elevation of the surface (a free surface or an arbitrary continuous surface of markers moving with the fluid) above a line y=constant. The Euler equations for the fluid and surface elevation are

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial x} \tag{1}$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial y} - g \tag{2}$$

$$\frac{\partial \eta}{\partial t} + u\,\frac{\partial \eta}{\partial x} = v \tag{3}$$

We now define x and t derivatives at a fixed value of x, but moving vertically with the surface. For arbitrary f(x,y,t), let the derivatives d/dx and d/dt be defined as follows:

$$\frac{df(x,\eta,t)}{dx} \equiv \frac{\partial f}{\partial x} + \frac{\partial \eta}{\partial x}\,\frac{\partial f}{\partial y} \tag{4}$$

$$\frac{df(x,\eta,t)}{dt} \equiv \frac{\partial f}{\partial t} + \frac{\partial \eta}{\partial t}\,\frac{\partial f}{\partial y} \tag{5}$$

$$= \frac{\partial f}{\partial t} + \left(v - u\,\frac{\partial \eta}{\partial x}\right)\frac{\partial f}{\partial y}$$

From this point on, derivatives d/dx and d/dt will implicitly take $y=\eta$, as in (4) and (5).

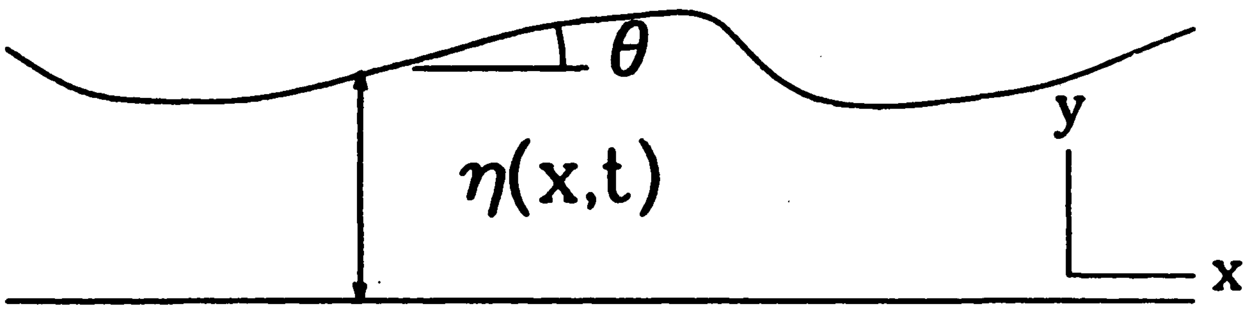

Figure 1 - Coordinate system and surface elevation referred to y=constant.

At the surface, (1) - (3) become

$$\frac{du}{dt} + u\,\frac{du}{dx} = -\frac{1}{\rho}\left(\frac{dp}{dx} - \frac{\partial\eta}{\partial x}\frac{\partial p}{\partial y}\right) \tag{6}$$

$$\frac{dv}{dt} + u\,\frac{dv}{dx} = -\frac{1}{\rho}\,\frac{\partial p}{\partial y} - g \tag{7}$$

$$\frac{\partial\eta}{\partial t} + u\,\frac{\partial\eta}{\partial x} = v\ . \tag{8}$$

Invoking

$$\frac{\partial}{\partial t}\frac{\partial\eta}{\partial x} = \frac{\partial}{\partial x}\frac{\partial\eta}{\partial t}\ , \tag{9}$$

and ρ = constant, (6) - (8) give straightforwardly

$$\frac{d}{dt}\left(u + \frac{\partial\eta}{\partial x}\,v\right) = -\frac{d}{dx}\left(\frac{u^2-v^2}{2} + u\,v\,\frac{\partial\eta}{\partial x} + \frac{p}{\rho} + g\eta\right). \tag{10}$$

In the more general case $\rho = \rho(p)$, p/ρ in (10) becomes $\int dp/\rho$. The general result depends neither on irrotationality nor incompressibility. The operand of the time derivative in (10) can be expressed as follows. Let v_T denote the tangential component of the surface velocity.

Then

$$u + \frac{\partial\eta}{\partial x}\,v = v_T\ \sec\,\theta, \tag{11}$$

with tan θ the surface slope (see Figure 1). Thus the left hand side of (11) is proportional to the tangential surface velocity.

B. Connection to Bernoulli's Law

For irrotational flows Bernoulli's Law is:

$$\frac{\partial\phi(x,y,t)}{\partial t} = -\left[\frac{1}{2}(u^2 + v^2) + \frac{p}{\rho} + gy + F(t)\right] \tag{12}$$

where $\phi(x,y,t)$ is the velocity potential and $F(t)$ is an arbitrary function of the time. By using (5), (12) becomes

$$\frac{d\phi(x,\eta,t)}{dt} \equiv \frac{d\phi_s}{dt} = -\left[\frac{1}{2}(u^2 + v^2) + \frac{p}{\rho} + gy + F(t)\right] + v\,\frac{\partial\eta}{\partial t}. \tag{13}$$

Now take d/dx of (13) and interchange d/dx and d/dt:

$$\frac{d}{dt}(d\phi_s/dx) = -\frac{d}{dx}\left[\frac{1}{2}(u^2 + v^2) + \frac{p}{\rho} + g\eta - v\,\frac{\partial\eta}{\partial t}\right]. \tag{14}$$

The kinematic condition along $y = \eta$ is:

$$v(x,\eta,t) \equiv v_s = \partial\eta/\partial t + u_s\partial\eta/\partial x, \tag{15}$$

so that the substitution of (15) for $\partial\eta/\partial t$ in (14) gives:

$$\frac{d}{dt}(d\phi_s/dx) = -\frac{d}{dx}\left[\frac{1}{2}(u_s^2 + v_s^2) + \frac{p_s}{\rho_s} + g\eta - u_s v_s\,\frac{\partial\eta}{\partial x}\right]. \tag{16}$$

Using (4), we interpret $d\phi_s/dx$ as $u_s + (\partial\eta/\partial x)v_s$, and so (16) is just (10). Consequently, we may interpret (10) as a Bernoulli Law. Note, however, that the starting point (12) is valid only for irrotational flows, while the result (10) is valid for rotational flow as well. Of course, the starting point for (10) is the set of Euler Equations, and so (10) is strictly valid only for flows in inviscid fluids.

3. Applications and Discussion

A. Sample Calculations using Conservation of Velocity.

The velocity conservation law represented by (10) is one of two general linear conservaton laws for two dimensional inviscid flows. The other is the well known conservation of mass:

$$\frac{\partial A}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{17}$$

where A is the cross-sectional area above still water level $\int \eta dz$, where z is the coordinate perpendicular to the plane of fluid motions, and F is the volume flux $\iint u\, dy\, dz$, taken over the whole fluid at a fixed value of x. Various forms of the Boussinesq Equations for irrotational water waves conserve two linear quantities, corresponding to the mass of (17) and the velocity of (10).

Simplifications to the Boussinesq Equations that permit waves to travel in one direction only, such as that derived by Korteweg and de Vries (1895), conserve neither the mass of (17) nor the velocity of (10). Boussinesq (1872), Miles (1979), and a few researchers in the intervening century recognized that mass was not conserved in monodirectional formulations, and added a reflected wave to balance a mass budget. The energetic part of the disturbance in the monodirectional formulation (a modified KdeV Equation in Miles's work) was left alone.

Witting (1982) has developed a model that incorporates mass and velocity conservation explicitly. The model equations of motion are:

$$\frac{d\eta}{dt} + \frac{1}{b}\frac{d}{dx}\left[\bar{u}(h_o+\eta)b\right] = 0 \tag{18}$$

$$\frac{dq}{dt} + \frac{d}{dx}\left[g\eta + \frac{1}{2}q^2 - \frac{1}{2}\left(1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right)^{-1}\left(\frac{\partial\eta}{\partial t} + q\frac{\partial\eta}{\partial x}\right)^2\right] = 0\ , \tag{19}$$

where

$$q = u_s + v_s\frac{\partial\eta}{\partial x}\ , \tag{20}$$

and

$$\bar{u} = \frac{1}{h_o+\eta}\int_{-h_o}^{\eta} u\, dy \tag{21}$$

The system of equations (18)-(21) is closed by relating $\bar{u}$ to q through power series expansion. Equation (18) is a restatement of (17), and (19) follows from (10) and (20). Some calculations using the theory and numerical methods described by Witting (1982) were designed to model experiments of Chang, et al. (1979) in a converging/diverging channel.

Figure 2 displays a profile $\eta = \eta\ (x,t_0)$ of a wave in a converging channel. In the figure we mark the crest location by $x = 0$, and express η and x in units of still water depth h_o. The particular time t_0 is chosen to match the time at which our crest passed the location of a wave probe of Chang, et al., who measure $\eta = \eta\ (x_0,t)$. The probe record is chosen from Chang, et al.'s Figure 7c. The profile of η in Figure 2 resembles the experimental record. Both are asymmetrical, about to the same degree, if allowance is made for Chang's slightly noisy records and for the small differences that must arise in comparing a space record with a time record. In order to identify the source of the asymmetry, we also plot q [nondimensionalized by $(gh_o)^{1/2}$], $(\eta+q)/2$, and $5(\eta-q)$ in Fig. 2. If all disturbances were linear long waves, $(\eta+q)/2$ would represent a disturbance traveling to the right (η in phase with q), and $5(\eta-q)$ would represent a disturbance traveling to the left (η in phase with $-q$, and scale expanded tenfold). In Fig. 2 the $(\eta+q)/2$ profile is still slightly asymmetrical, which agrees with the theoretical profile that Chang, et al. derive from a Korteweg-deVries theory in comparing their experimental data to theory. The $5(\eta-q)$ profile clearly shows a reflected wave, which the Korteweg-deVries theory does not describe. (We should note that only for $x \lessapprox -4$ is the $(\eta+q)/2$ profile small enough that $5(\eta-q)$ unambiguously signals a reflected wave. For $x \gtrapprox -4$ a decomposition into right- and left-going waves needs a more detailed theory than linear long wave theory.)

If mass conservation is added as in Miles (1979), the reflected wave will be taken into account, perhaps adequately.

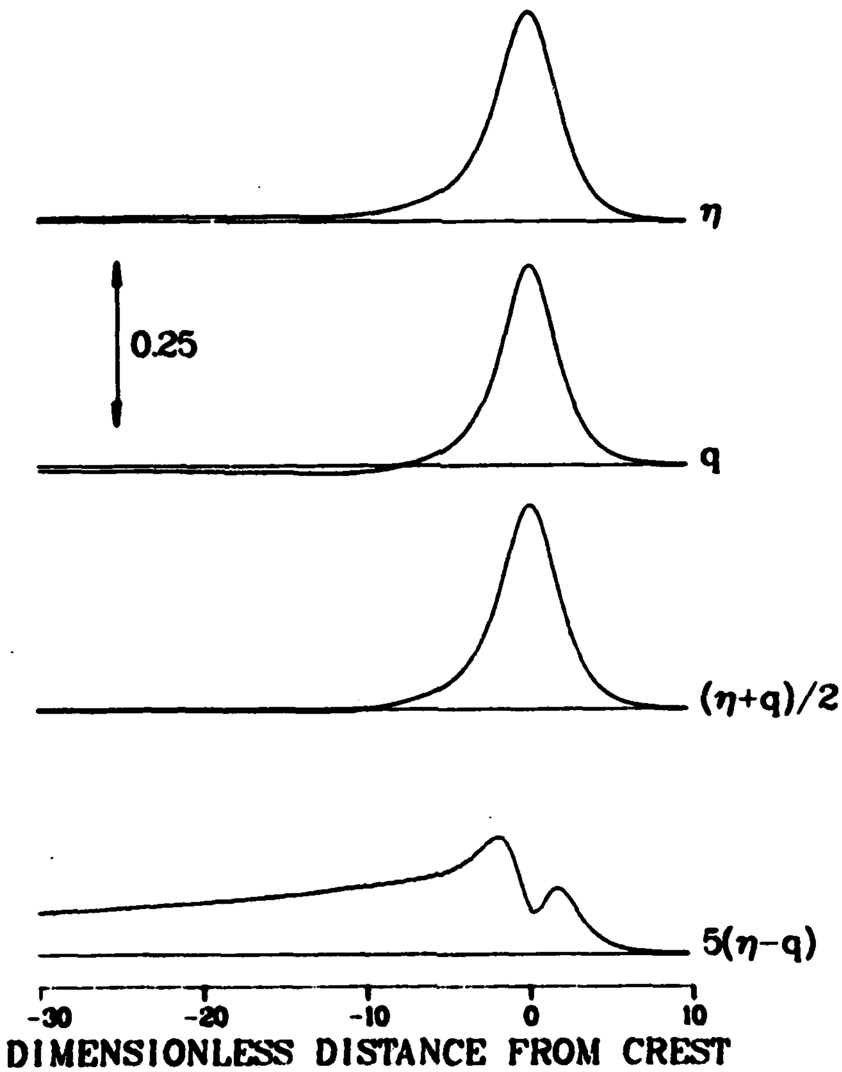

Figure 2 - Solitary wave in a converging channel. The geometry is set to that of of Chang, et al. (1979), which has straight walls converging from the end of an entrance section at an angle of 1.074°. The elevation above still water level and distance from the crest are measured in units of h_o, the initial depth (=0.40m). The velocity variable $q \equiv u_s + \eta' v_s$ is measured in units of $(gh_o)^{1/2}$. At the crest the channel is 0.434 as broad as the entrance setion, where the channel walls are parallel. Initial conditions were set to produce a solitary wave in the entrance section having nondimensional amplitude 0.197. The crest in the figure has nondimensional elevation 0.321.

Figure 3 displays profiles of η, q, $(\eta+q)/2$ and $5(\eta-q)$ in a diverging channel. Again x = 0 marks the crest location. The time at which the profile "snapshots" are taken matches the time that the crest passed one of the wave probes of Chang, et al. (the probe record is shown in the fifth display of their Figure 6). The computed profile of η in Figure 3 again reproduces the essential features of the experimental record. The wave is asymmetrical in the oppsite sense of Figure 2, and has an oscillatory tail that is mostly negative. By contrast, the Korteweg-deVries theory of Chang, et al., which conserves neither mass nor velocity, gives an almost symmetrical solitary wave with no reflected wave and no tail. From the bottom curve of Figure 3 we see that some of the disturbance behind the solitary wave is a reflected wave of depression. From the $(\eta+q)/2$ curve we see that the rest of the disturbance behind the solitary wave is an oscillatory wave, mostly negative, that is traveling toward the right, i.e. along with the solitary wave.

Thus, the calculations displayed in Fig. 3, which conserve both mass and velocity, reproduce the features behind the solitary wave that show up in the laboratory experiments. These features appear to be the result of waves traveling both to the left and to the right. We believe that the rightward traveling wave is composed of both a modified solitary wave and a re-reflected tail akin to that described by Witting (1981).

B. Discussion

The principal new result of this research is the conservation of velocity law derived under general circumstances (Eq. 10). It can be used explicitly in numerical calculations, or can be used as a test for the accuracy of calculations that do not make use of it explicitly. Because it is embedded in the Euler Equations, direct use of these equations should conserve it automatically. Certain approximations, such as the modified Korteweg-deVries

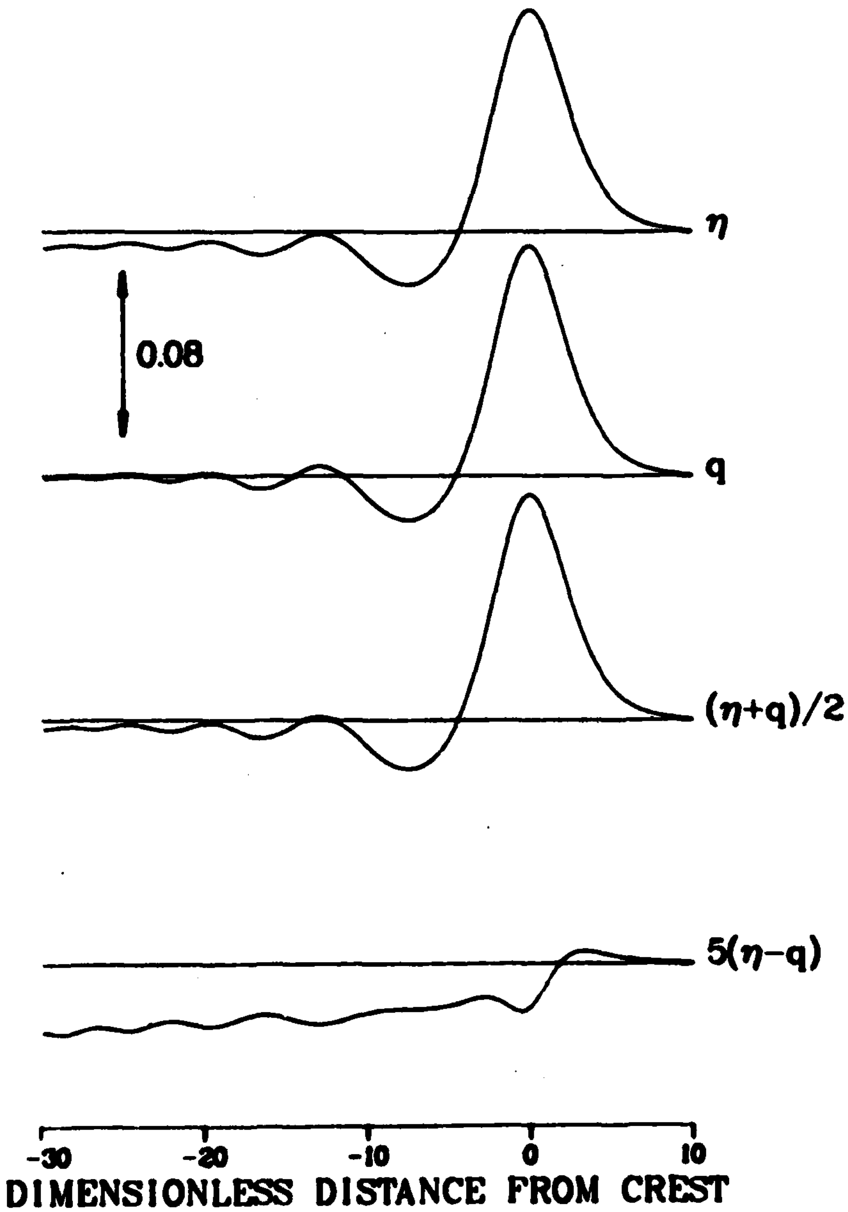

Figure 3 - Solitary wave in a diverging channel. The geometry is set to that of Chang, et al., (1979), which has straight walls diverging from the end of an entrance section at an angle of 1.074°. As in Fig. 2, h_o and $(gh_o)^{1/2}$ are the units of length and velocity (but with h_o = 0.30m here). At the crest the channel is 6.822 times broader than the entrance section. In the entrance section the distrubance was a solitary wave of amplitude 0.386. The crest in the figure has nondimensional elevation 0.109.

Equation for wave propagation in a channel of varying section, violate conservation of mass and velocity. A model intended to account for a broader range of phenomena as described above should conserve both mass and velocity. Equation (10) with (17) provides an explicit (though not unique) mechanism for performing this task.

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